KIRCHHOFF PLATE MODELLING USING FINITE ELEMENT METHOD

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Dedicated to my beloved,
Abang,
dearest Mak and Ayah,
my brothers, Abe Yie, Abe We, Abe Pan,
Ran, Acah and Aqil,

&

my supervisor,

Prof. Dr. Shaharuddin Salleh

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ABSTRACT

The Kirchhoff plate theory works well for thin plates where the real shear strains are small. In this study, the development of Kirchhoff plate theory using FEM is presented. The equilibrium condition of the problem defined as $\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q = 0$ is investigated in providing the appropriate boundary conditions, hence to the establishment of the FE formulation of the problem. The plate elements developed are the two-dimensional triangular element. To meet the convergence criteria, the quadratic interpolation function is adopted and the six nodes triangular element is developed. The deflection w takes the form of $w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$. The numerical results of two neighbouring six nodes triangular elements are studied. These elements are considered to be interconnected at specified nodes which lie on the element boundaries where adjacent elements are considered to be connected. In each piece or element, the element shape function N_i, the stiffness matrix $\mathbf{K}$, and the load vector $\mathbf{f}_l$ are derived. The assemblage of these matrices together with the derivation of boundary vector $\mathbf{f}_b$ will yield to an approximate solution for the displacement of the problem. The computational scheme is developed by using Matlab programming language on the Windows environment for computing the problem studied.

ABSTRAK

Teori plat Kirchhoff berjalan dengan baiknya untuk plat-plat nipis di mana ricihan ketegangan sebenar adalah kecil. Dalam kajian ini, pembangunan teori plat Kirchhoff menggunakan FEM dibentangkan. Keadaan keseimbangan masalah yang didefinisikan sebagai $\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q = 0$ diselidiki dalam menyediakan syarat-syarat sempadan yang bersesuaian , seterusnya untuk penubuhan perumusan FE masalah tersebut. Elemen-elemen plat yang dikaji adalah berunsurkan segitiga dua dimensi. Dalam menepati kriteria penumpuan, fungsi interpolasi kuadratik dipilih dan unsur segitiga enam nodus dibangunkan. Pesongan w mengambil bentuk sebagai $w(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$. Penyelesaian berangka dua segitiga enam nodus yang berjiran adalah dikaji. Unsur-unsur ini dianggap saling berkait pada nodus-nodus yang ditetapkan yang mana nodus-nodus ini berada di garisan sempadan unsur-unsur berjiran tersebut. Dalam setiap bahagian atau unsur tersebut, fungsi bentuk unsur N_i, matriks kekukuhan $\mathbf{K}$, dan vektor beban $\mathbf{f}_l$ diterbitkan. Himpunan matrik-matrik ini bersama dengan penerbitan vektor sempadan $\mathbf{f}_b$ akan menghasilkan satu penyelesaian hampiran kepada masalah yang dikaji. Perisian berangka untuk menyelesaikan masalah yang dikaji dibangunkan dengan menggunakan bahasa pengaturcaraan Matlab dan diaplikasikan pada persekitaran Windows.

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LIST OF SYMBOLS

D	-	flexural rigidity of a plate
t	-	plate thickness
q	-	transverse loading of the plate
w	-	deflection of the plate
σ_{ij}	-	stress component
M_{ij}	-	moment component
V_{ij}	-	vertical force component
N_{ij}	-	horizontal force component
u^0	-	displacement of the mid-plane in the x-directions
υ^0	-	displacement of the mid-plane in the y-directions
ε_{ij}	-	strain component
γ_{ij}	-	shear strain component
$\boldsymbol{\sigma}$	-	stress components matrix
$\boldsymbol{\varepsilon}$	-	strain components matrix
$\mathbf{D}$	-	plane stress constitutive matrix
E	-	Young's modulus coefficient
v	-	Poisson's ratio coefficient
$\boldsymbol{\kappa}$	-	curvature matrix
$\mathbf{M}$	-	moments matrix
$\overset{*}{\nabla}$	-	matrix differential operator

$\mathbf{n}$	-	a unit normal vector located in the xy-plane
$\mathbf{m}$	-	a unit vector that is orthogonal to $\mathbf{n}$
$\mathbf{t}$	-	traction vector
$\mathbf{R}$	-	square matrix
$\mathbf{r}$	-	unit vector defined in the xy-plane
α	-	Parameter
β	-	Parameter
$\nabla \upsilon$	-	the gradient of υ
ϕ	-	two-dimensional quantity [$\phi(x, y)$]
A	-	region on mid-plane at two-dimensional problem
$\mathscr{L}$	-	boundary of A
θ_n	-	slope of a straight line normal to $\mathscr{L}$
θ_m	-	slope of a straight line tangential to $\mathscr{L}$
$\upsilon(x, y)$	-	weight function
N_i	-	the element shape function
u_i	-	nodal values
$\mathbf{c}$	-	parameter
$\mathbf{K}$	-	the stiffness matrix
$\mathbf{f}_b$	-	the boundary vector
$\mathbf{f}_l$	-	the load vector
$\mathbf{f}$	-	the force vector
L_i	-	triangular coordinate system $(i = 1,2,3)$
A	-	area of the triangle
$\mathbf{N}$	-	element shape function matrix

LIST OF APPENDICES

CHAPTER I

RESEARCH FRAMEWORK

1.1 Introduction

A variety of specializations under the umbrella of the mechanical engineering discipline such as aeronautical, biomechanical, and automotive industries are modelled by differential equations. Usually, the problem addressed is too complicated to be solved by classical analytical methods. The finite element method (FEM) is a numerical approach by which general differential equations can be solved in an approximate manner. In other words, FEM is an approximate numerical procedure for analyzing large structures and continua (Cook *et al*., 1989). Figure 1.1 illustrates generally how the physical phenomenon encountered in engineering mechanics is modelled.

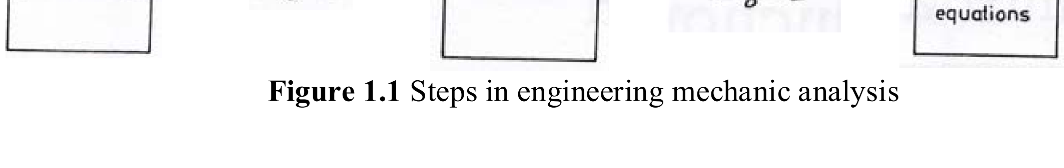

Figure 1.1 Steps in engineering mechanic analysis

As the FEM is a numerical, means of solving general differential equations, it can be applied to various physical phenomena. Furthermore, FEM became popular with the advancements in digital computers since they allow engineers to solve large systems of equations quickly and efficiently. The method becomes a very useful tool for the solution of many types of engineering problems such as the analysis of the plate and beam structures, heat transfer and fluid flow. The method is also widely used in the design of air frames, ships, electric motors, heat engines and spacecraft.

Although the finite element model does not behave exactly like the actual physical structure, to obtain sufficiently accurate results for most practical applications become possible. In FEM, the finite element model is created by dividing the structure into smaller parts, called finite elements. Each element is interconnected by nodes and the selection of elements for modelling the structure depends upon the behavior and geometry of the structure being analyzed. The modelling pattern, which is generally called mesh, is a very important part of the modelling process. This is because; the results obtained depend upon the selection of the finite elements and the mesh size. After having determined the behavior of all elements, these elements are then patched together to form the entire region, which enable to obtain an approximate solution for the behavior of the entire body. The situation discussed is shown in Figure 1.2 while Figure 1.3 shows the finite element mesh of the structural part of a car.

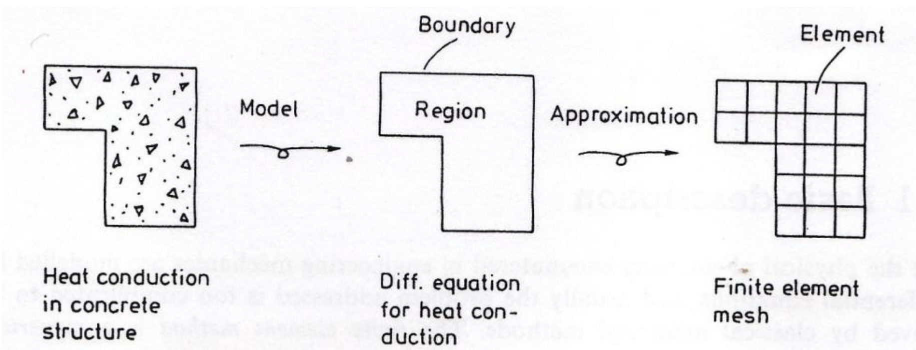

Figure 1.2 Illustration of modelling steps

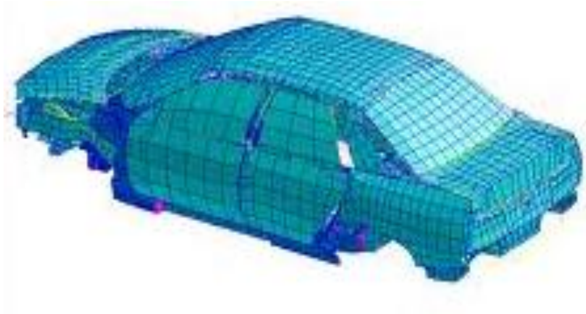

Figure 1.3 Finite element mesh of the structural part of a car

As mentioned in the second paragraph before, one of the applications of FEM is the formulation of plate elements. Plate elements can be formulated and modelled mathematically based on the Kirchhoff plate theory. The focus of this dissertation is to develop the triangular elements for the finite element analysis of Kirchhoff plate problem. An important aspect of the work is to implement the problem on the computer using Matlab programming language. Figure 1.4 shows a region of thin plate is divided into finite elements.

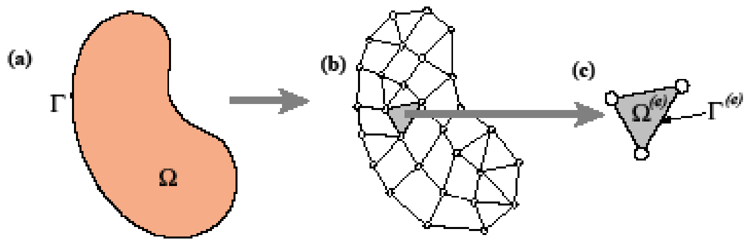

Figure 1.4 A thin plate subdivided into finite elements

1.2 Problem Statement

The subject of plates was one of the first to which the finite element method was applied in the early of 1960's. At that time the various difficulties that were to be encountered were not fully appreciated and the topic remains one in which research is active to the present day. The first convincing plate theory was established by Kirchhoff which therefore also termed Kirchhoff plate theory as described by Boresi *et al.* (1978) and Timoshenko and Woinowsky-Krieger (1959). In this study, it will be concentrated in deriving a numerical solution for plate problem (for triangular elements), given their boundary conditions by using finite element method. Great effort also will be concentrated in developing the computational scheme of the problem by using Matlab programming language.

1.3 Objectives of the Study

The objectives of this study are:

1. To study the various aspects of plate theory and its finite element (FE) formulation.

2. To set up a numerical scheme by using FEM in solving the Kirchhoff plate problem.

3. To develop a computational scheme of Kirchhoff plate problem by using Matlab programming language.

1.4 Scope of the Study

The dissertation will be focusing on the formulation by using FEM for 2-dimensional problem. The simplest plate element for the analysis of plates of arbitrary shape which is the six nodes triangular element mesh with quadratic interpolation functions is considered in this study. Both numerical and the computational scheme of the problem then will be carried out. Effort will be concentrated on developing the computational scheme/simulation of the problem by using Matlab programming language.

1.5 Significance of the Study

The significance of the study is stated as follows:

1. The derivation of numerical codes and efficient algorithms of the Kirchhoff plate problem help to solve the related problems in the future. The results hence will contribute towards an enhanced understanding of the problem.

2. The simulation of the problem gives a significant results and solutions for validation purposes in related problems.

1.6 Research Methodology

In this study, there are five steps that will be concentrated in order to get the computational scheme for six nodes triangular Kirchhoff plate problem. The steps are:

1. Literature review on the Kirchhoff plate theory.

2. Comprehend the various aspects of plate theory. This covers equilibrium conditions, kinematic relations, constitutive relation, differential equations, and boundary conditions.

3. Next, the derivation of the differential equations of plate theory will be conducted. Hence to the establishment of the FE formulation of the Kirchhoff plate theory.

4. Then, the numerical solution of the problem is carried out.

5. Lastly, the simulation of the problem is presented. The computational code is developed by using Matlab programming language and running on the windows environment.

1.7 Thesis Organization

The dissertation is organized into six chapters. Chapter I is the research framework. This chapter describes in detail some discussion with the introduction of the study, a description of the problem, the objectives of the study, scope of the study, significance of the study, research methodology, and chapter organization.

Chapter II starts with a brief literature review on the development of plate theory. This chapter also contains a review and discussion the various aspects of plate theory. The establishment of the differential equations of the problem is also presented in this chapter.

Chapter III discusses in detail the FE formulation of the Kirchhoff plate theory. It starts with the derivation of the weak form of the problem, and followed with the establishment of stiffness matrix, the boundary vector, and the load vector.

Chapter IV presents the numerical results of the Kirchhoff plate problem. The process of the FEM in obtaining the numerical scheme is outlined in detail in this chapter.

In Chapter V, the computational scheme of the problem is presented. This chapter also highlights on the analysis and discussion of the simulation model developed with Matlab programming language.

Lastly, we will make some conclusions of this study in Chapter VI. This chapter presents a summary of the important results and a discussion of the results. Suggestions for future research are also given in this chapter. All the references quoted are listed in the reference section after this chapter.